

Kaluza-Klein magnetic monopole a la Taub-NUT

Nematollah Riazi* and S. Sedigheh Hashemi

Department of Physics, Shahid Beheshti University, G.C., Evin, Tehran 19839, Iran

August 21, 2015

Abstract

We present a Taub-NUT Kaluza-Klien vacuum solution and using the standard Kaluza-Klein reduction, show that this solution is a static magnetic monopole in 3+1 dimensional spacetime. We find that the four dimensional matter properties do not obey the equation of state of radiation and there is no event horizon. A comparison with the available magnetic monopole solutions, the nature of the singularity, and the issue of vanishing or negative mass are discussed.

1 Introduction

One of the oldest ideas that unify gravity and electromagnetism is the theory of Kaluza and Klein which extends space-time to five dimensions [1]. The physical motivation for this unification is that the vacuum solutions of the $(4+1)$ Kaluza-Klein field equations reduce to the $(3+1)$ Einstein field equations with effective matter and the curvature in $(4+1)$ space induces matter in $(3+1)$ dimensional space-time [2]. With this idea, the four dimensional energy-momentum tensor is derived from the geometry of an exact five dimensional vacuum solution, and the properties of matter such as density and pressure as well as electromagnetic properties are determined by such a solution. In other words, the field equations of both electromagnetism and gravity can be obtained from the pure five-dimensional geometry.

Kaluza's idea was that the universe has four spatial dimensions, and the extra dimension is compactified to form a circle so small as to be unobservable. Klein's contribution was to make a reasonable physical basis for the compactification of the fifth dimension [3]. This school of thinking later led to the eleven-dimensional supergravity theories in 1980s and to the "theory of everything" or ten-dimensional superstrings [4].

Many spherically symmetric solutions of Kaluza-Klein type are investigated in [5]-[6]. In the work by Gross and Perry [7] and Davidson and Owen [8] some other solutions of the Kaluza-Klein equations are discussed. The $(4+1)$ analogues of $(3+1)$ Schwarzschild solution are among these solutions. There are also solutions of Kaluza-Klein equations which do not have event horizon of the type which exists in the Einstein's theory.

In this paper, we present a vacuum solution of Kaluza-Klein theory in five-dimensional space-time which is closely related to the Taub-NUT metric. The Taub-NUT solution has many interesting features; it carries a new type of charge (NUT charge) which has topological origins and can be regarded as "gravitational magnetic charge", so the solution is known in some other contexts as the Kaluza-Klein magnetic monopole [9]

*Electronic address: n_riazi@sbu.ac.ir

The plan of this paper is as follows. In section 2, we briefly discuss the formalism of five dimensional Kaluza-Klein theory and the effective four-dimensional Einstein-Maxwell equations. We will then present a Taub-NUT-like Kaluza-Klein solution and investigate its physical properties in four dimensions in section 3. In the last section we will draw our main conclusions.

2 Kaluza-Klein theory

Kaluza (1921) and Klein (1926), used one extra dimension to unify gravity and electromagnetism in a theory which was basically five-dimensional general relativity [10]. The theoretical elegance of this idea is revealed by studying the vacuum solutions of Kaluza-Klein equations and the matter induced in the four-dimensional spacetime[11]. Thus, we are chiefly interested in the vacuum five dimensional Einstein equations. For any vacuum solution, the energy-momentum tensor vanishes and thus $\hat{G}_{AB} = 0$ or, equivalently $\hat{R}_{AB} = 0$, where $\hat{G}_{AB} \equiv \hat{R}_{AB} - \frac{1}{2}\hat{R}\hat{g}_{AB}$ is the Einstein tensor, \hat{R}_{AB} and $\hat{R} = \hat{g}_{AB}\hat{R}^{AB}$ are the five-dimensional Ricci tensor and scalar, respectively. \hat{g}_{AB} is the metric tensor in five dimensions. Here, the indices A, B, \dots run over 0...4 [4].

Generally, one can identify the $\mu\nu$ part of \hat{g}^{AB} with $g^{\mu\nu}$, which is the contravariant four dimensional metric tensor, A_μ with the electromagnetic potential and ϕ as a scalar field. The correspondence between the above components is

$$\hat{g}^{AB} = \begin{pmatrix} g^{\mu\nu} & -\kappa A^\mu \\ -\kappa A^\nu & \kappa^2 A^\sigma A_\sigma + \phi^2 \end{pmatrix} \quad (1)$$

where κ is a coupling constant for the electromagnetic potential A^μ and the indices μ, ν run over 0...3.

The five dimensional field equations reduce to the four dimensional field equations [12],[13]

$$G_{\mu\nu} = \frac{\kappa^2 \phi^2}{2} T_{\mu\nu}^{EM} - \frac{1}{\phi} [\nabla_\mu (\partial_\nu \phi) - g_{\mu\nu} \square \phi], \quad (2)$$

and

$$\begin{aligned} \nabla^\mu F_{\mu\nu} &= -3 \frac{\partial^\mu \phi}{\phi} F_{\mu\nu}, \\ \square \phi &= \frac{\kappa^2 \phi^3}{4} F_{\mu\nu} F^{\mu\nu}, \end{aligned} \quad (3)$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}^{EM} \equiv \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - F_\mu^\sigma F_{\nu\sigma}$ is the electromagnetic energy-momentum tensor and the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Knowing the five dimensional metric, therefore, leads to a complete knowledge of the four dimensional geometry, as well as the electromagnetic and scalar fields.

3 Taub-NUT solution and Kaluza-Klein magnetic monopole

The Taub-NUT solution was first discovered by Taub (1951), and subsequently extended by Newman, Tamburino and Unti (1963) as a generalization of the Schwarzschild spacetime [14]. This solution is a single, non-radiating extension of the Taub universe. It is an anisotropic but

spatially homogeneous vacuum solution of Einstein's field equations with topology $R^1 \times S^3$. The Taub metric is given by

$$ds^2 = -\frac{1}{V(t)}dt^2 + 4b^2V(t)(d\psi + \cos\theta d\phi)^2 + (t^2 + b^2)(d\theta^2 + \sin^2\theta d\phi^2), \quad (4)$$

where $V(t) = -1 + 2(mt + b^2)(t^2 + b^2)^{-1}$, m and b are positive constants, ψ, ϕ, θ are Euler angles with usual ranges [15]. The Taub-NUT solution is nowadays being used in the context of higher-dimensional theories of semi-classical quantum gravity [16]. As an example, in the work by Gross and Perry [7] and of Sorkin [17], soliton solutions were obtained by embedding the Taub-NUT gravitational instanton inside the five dimensional Kaluza-Klein manifold[14]. Solitons are static, localized, and non-singular solutions of nonlinear field equations which resemble particles. One such solution which obeys the Dirac quantization condition is considered in [7].

The Kaluza-Klein monopole of Gross and Perry is described by the following metric which is a generalization of the self-dual Euclidean Taub-NUT solution [7]

$$ds^2 = -dt^2 + V(dx^5 + 4m(1 - \cos\theta)d\phi)^2 + \frac{1}{V}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2), \quad (5)$$

where

$$\frac{1}{V} = 1 + \frac{4m}{r}. \quad (6)$$

This solution has a coordinate singularity at $r = 0$ which is called NUT singularity. This can be absent if the coordinate x^5 is periodic with period $16\pi m = 2\pi R$, where R is the radius of the fifth dimension. Thus $m = \frac{\sqrt{\pi G}}{2e}$ [18]. The gauge field A_ν is given by $A_\phi = 4m(1 - \cos\theta)$, and the magnetic field is $B = \frac{4m\mathbf{r}}{r^3}$, which is clearly that of a monopole and has a Dirac string singularity in the range $r = 0$ to ∞ . The magnetic charge of this monopole is $g = \frac{m}{\sqrt{\pi G}}$ which has one unit of Dirac charge. In this model, the total magnetic flux is constant. For this solution, the soliton mass is determined to be $M^2 = \frac{m_p^2}{16\alpha}$ where m_p is the Planck mass. Gross and Perry showed that the Kaluza-Klein theory can contain magnetic monopole solitons which would support the unified gauge theories and allow for searching the physics of unification.

Here, we introduce a metric which is a vacuum five dimensional solution, having some properties in common with the monopole of Gross and Perry, while some other properties being different. The proposed static metric is given by

$$ds_{(5)}^2 = -dt^2 + w(r)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) + \frac{k}{w(r)}(d\psi + Q\cos\theta d\phi)^2. \quad (7)$$

Here, the extra coordinate is represented by ψ . k and Q are constants and $w(r)$ is a function (to be determined) of the radial coordinate r . The coordinates take on values within the usual ranges: $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 2\pi$.

The Ricci scalar associated with the five dimensional metric (7) is given by

$$R = \frac{1}{2} \frac{1}{w(r)^3 r^4} [2r^4 w(r) w''(r) + 4w'(r) w(r) r^3 - w'(r)^2 r^4 + kQ^2]. \quad (8)$$

Since we are interested in vacuum solution where the Ricci tensor is zero ($R_{AB} = 0$) it is necessary but not sufficient to have a zero Ricci scalar $R = 0$ which can be solved for the function $w(r)$ to give

$$w(r) = k_1 + \frac{k_2}{r} + \frac{k_3}{r^2}, \quad (9)$$

where k_1 , k_2 and k_3 are constants. By substituting the function $w(r)$ in the Ricci scalar (8) and requiring a zero Ricci scalar, one can obtain the following constraint between the constants

$$kQ^2 + 4k_1k_3 - k_2^2 = 0, \quad (10)$$

or

$$k_3 = \frac{1}{4k_1} (k_2^2 - kQ^2). \quad (11)$$

Thus, the Ricci scalar in terms of the above constants reduces to

$$R = \frac{1}{2} \frac{r^2(4k_1 - 1)(kQ^2 - k_2^2)}{(kQ^2 - k_2^2 - k_2r - k_1r^2)^3}. \quad (12)$$

In order to have a zero Ricci scalar the two following possibilities remain

$$R = 0 \implies \begin{cases} k_1 = \frac{1}{4} \implies R_{AB} \neq 0, \\ \text{or} \\ k_2 = \pm\sqrt{k}Q \implies R_{AB} = 0, \quad R_{ABCD} \neq 0. \end{cases} \quad (13)$$

As it is seen from Eq. (13) with $k_1 = \frac{1}{4}$, the Ricci tensor is not zero and does not give a vacuum solution, thus this case is discarded. However, the second choice $k_2 = \pm\sqrt{k}Q$ gives a Ricci flat solution with a non-zero Riemann tensor in five dimensions which is what we are interested in. Consequently, the general form of the function $w(r)$ will be given by

$$w(r) = k_1 \pm \frac{\sqrt{k}Q}{r}. \quad (14)$$

In what follows, we take the constants $k_1 = 1$, $Q = 1$ and $k = 4m^2$. Therefore, metric (7) reduces to (we choose the minus sign for the second term in $w(r)$)

$$ds_{(5)}^2 = -dt^2 + \left(1 - \frac{2m}{r}\right) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + \left(\frac{4m^2}{1 - \frac{2m}{r}}\right) (d\psi + Q \cos \theta d\phi)^2. \quad (15)$$

The Killing vectors associated with metric (15) are given by

$$\begin{aligned} K_0 &= (1, 0, 0, 0, 0), \quad K_1 = (0, 0, 0, 0, 1), \quad K_2 = (0, 0, 0, 1, 0), \\ K_3 &= (0, 0, -\sin \phi, -\cot \theta \cos \phi, \csc \theta \cos \phi), \\ K_4 &= (0, 0, \cos \phi, -\cot \theta \sin \phi, \csc \theta \sin \phi), \end{aligned} \quad (16)$$

which are the same as in the Taub-NUT space discussed in [19], where the authors studied spinning particles in the Taub-NUT space.

The gauge field, A_μ and the scalar field ϕ deduced from the metric (15) with the help of (1) are

$$A_\phi = \frac{\cos \theta}{\kappa}, \quad (17)$$

and

$$\phi^2 = \frac{4m^2}{1 - \frac{2m}{r}}, \quad (18)$$

respectively. The only non-vanishing component for the electromagnetic field tensor which is related to the gauge field A_μ via $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is

$$F_{\theta\phi} = -F_{\phi\theta} = -\frac{\sin\theta}{\kappa}, \quad (19)$$

which corresponds to a radial magnetic field $B_r = \frac{1}{\kappa r^2}$ with a magnetic charge $Q_M = \frac{1}{\kappa}$. As the radial coordinate r goes to infinity $r \rightarrow \infty$, the scalar field equals $\phi_0^2 = 4m^2$ so that the second part of equation (2) becomes zero, therefore

$$G_{\mu\nu} = \frac{\kappa^2 \phi_0^2}{2} T_{\mu\nu}^{EM} = 8\pi G T_{\mu\nu}, \quad \text{as } r \rightarrow \infty, \quad (20)$$

where we have put the speed of light c equal to 1. By comparing this equation with the Einstein equation, we obtain $\frac{\kappa^2 \phi_0^2}{2} = 8\pi G$, thus the constant κ equals $\kappa = \frac{2}{m} \sqrt{\pi G}$. Straightforwardly, the total magnetic monopole charge is given by

$$Q_M = \frac{m}{2} \frac{1}{\sqrt{\pi G}}. \quad (21)$$

The total magnetic flux through any spherical surface centered at the origin is calculated as [20]

$$\Phi_B = \int F = \frac{1}{2} \oint F_{\mu\nu} d\Sigma^{\mu\nu} = \frac{\pi}{\kappa}, \quad (22)$$

where $F_{\mu\nu}$ is the electromagnetic field tensor and $d\Sigma^{\mu\nu}$ is an element of two-dimensional surface area. It is observed from Eq. (22) that the flux of the magnetic monopole is constant (i.e. we have a singular magnetic charge).

We now show that equation (3) is satisfied explicitly in four dimensions. We first note that

$$\square\phi = \frac{4m^3}{r^4(1 - \frac{2m}{r})^{\frac{7}{2}}}. \quad (23)$$

The value of the scalar $F_{\mu\nu}F^{\mu\nu}$ is easily calculated to be

$$F_{\mu\nu}F^{\mu\nu} = \frac{2}{\kappa^2 r^4 (1 - \frac{2m}{r})^2}, \quad (24)$$

and therefore

$$\frac{\kappa^2 \phi^3}{4} F_{\mu\nu}F^{\mu\nu} = \frac{4m^3}{r^4(1 - \frac{2m}{r})^{\frac{7}{2}}}, \quad (25)$$

which together with (23) shows that equation (3) is satisfied.

The four dimensional metric deduced from equation (15) with the use of (1) leads to the following asymptotically flat spacetime:

$$ds_{(4)}^2 = -dt^2 + \left(1 - \frac{2m}{r}\right) dr^2 + r^2 \left(1 - \frac{2m}{r}\right) [d\theta^2 + \sin^2\theta d\phi^2]. \quad (26)$$

For this metric, the Ricci scalar and the Kretschmann invariant K are given by

$$R = \frac{6m^2}{r^4 (1 - \frac{2m}{r})^3}, \quad (27)$$

and

$$K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{m^2}{(r-2m)^5} \left[\frac{2}{r-2m} + \frac{2r-3m}{r^2} \right]. \quad (28)$$

It is seen that the four dimensional metric (26) has two **curvature** singularities at $r = 0$ and $r = 2m$. In order to achieve more insight into the nature of the singularity at $r = 2m$, let us calculate the proper surface area of a S^2 hypersurface at constant t and r ;

$$A(r) = \int \sqrt{g^{(2)}} dx^2 = \int r^2 \left(1 - \frac{2m}{r} \right) \sin\theta d\theta d\phi = 4\pi r^2 \left(1 - \frac{2m}{r} \right). \quad (29)$$

It can be seen that the surface area $A(r)$ becomes zero at $r = 2m$, showing that the space shrinks to a point at $r = 2m$. This result shows that the $r = 2m$ hypersurface is not an event horizon. For the case $r > 2m$ the signature of the metric is proper $(-, +, +, +)$ but for the range $r < 2m$ the signature of the metric will be improper and non-Lorentzian $(-, -, -, -)$, thus the patch $r < 2m$ should be excluded from the spacetime. From now on, the spacetime is only considered in the range $r \geq 2m$. Since the range $0 < r < 2m$ is omitted from the spacetime, there is now only one curvature singularity at $r = 2m$. In order to see this more clearly, let us transform the metric (26) into the following form, using the radial coordinate transformation $\tilde{r}^2 = r^2 \left(1 - \frac{2m}{r} \right)$;

$$ds_{(4)}^2 = -dt^2 + \frac{\tilde{r}^4}{(m^2 + \tilde{r}^2)(m + \sqrt{m^2 + \tilde{r}^2})^2} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2. \quad (30)$$

While the metric (26) is singular at $r = 0$ and $r = 2m$, the transformed metric is singular only at $\tilde{r} = 0$ which is a curvature naked singularity.

The properties of the induced matter associated with the above metric can be gained by the equation (3). The components of the energy-momentum tensor can be easily calculated for the metric (26). We find that the effective source is like a fluid with an anisotropic pressure. At sufficiently large r , the energy-momentum components will tend to zero. The trace of the energy-momentum tensor does not vanish generally, which shows that the effective matter field around the singularity can not be considered as an ultra-relativistic quantum field in contrast to the Kaluza-Klein solitons described in [21]. The gravitational mass can be obtained from the energy-momentum tensor components according to

$$M_g(r) \equiv \int_{2m}^r (T_0^0 - T_1^1 - T_2^2 - T_3^3) \sqrt{g^{(3)}} dV_3, \quad (31)$$

where $g^{(3)}$ is the determinant of the 3-metric and dV_3 is a 3D volume element. One would expect the mass of the monopole be independent of the sign of the magnetic charge, since the electromagnetic energy density depends on the magnetic field squared and independent of its direction. Using the components of the T^μ_ν make the value of integrand zero so that

$$M_g(r) = 0, \quad (32)$$

meaning that the gravitational mass vanishes everywhere. The conserved Komar mass which is defined as [22]

$$M_g(r) = -\frac{1}{8\pi} \oint_S \nabla^\mu K^\nu dS_{\mu\nu}, \quad (33)$$

also leads to

$$M_g(r) = 0, \quad (34)$$

which is in agreement with the gravitational mass.

If in the process of compactification from $(4 + 1)$ to $(3 + 1)$ dimensions we use the ansatz

$$\hat{G}_{AB} = \phi^\beta \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix}, \quad (35)$$

then the choice of β for which ϕ does not appear explicitly is called the Einstein frame. Using the above equation will lead to the four dimensional metric

$$ds_{(4)}^2 = \phi^{-\beta} \left[-dt^2 + \left(1 - \frac{2m}{r}\right) dr^2 + r^2 \left(1 - \frac{2m}{r}\right) d\Omega^2 \right], \quad (36)$$

by choosing $\phi^{-\beta} = (1 - \frac{2m}{r})^{-\frac{1}{2}}$, equation (36) reduces to

$$ds^2 = -(1 - \frac{2m}{r})^{-1/2} dt^2 + (1 - \frac{2m}{r})^{1/2} (dr^2 + r^2 d\Omega^2). \quad (37)$$

This is the metric obtained by Gibbons and Manton in [23] by replacing m with $-m$. The metric (37) with $-m$ gives a finite value for total energy which is equal to $E = \frac{m}{2G}$.

4 Conclusion

Inspired by the Taub-NUT solution, we introduced a Kaluza-Klein vacuum solution in $(4 + 1)D$, which described a magnetic monopole in $(3 + 1)D$ spacetime. The gravitational mass of the solution was shown to vanish, contrary to the magnetic monopole of Gibbons and Manton[23]. The fluid supporting the four dimensional space-time was shown to differ from that of an ultra-relativistic fluid, in contrast with the work by Wesson and Leon[21]. The pressure is anisotropic in both works.

The main conclusion of this paper is that the presented Taub-NUT Kaluza-Klein solution contains a singular magnetic monopole with vanishing gravitational mass. We calculated the magnetic charge and showed that the total magnetic flux of the monopole through any spherical surface centered at the origin was constant, indicating that the monopole is singular and there is no extended magnetized fluid. The gravitational mass was derived in two different ways and was shown to vanish using both definitions. It was pointed out that the singularity which appears at finite r is neither a horizon, nor a surface of finite, non-vanishing surface area. In a more appropriate coordinate system, this was shown to be curvature singularity at $\tilde{r} = 0$.

Acknowledgements

The authors acknowledge the support of Shahid Beheshti University Research Council.

References

- [1] T. Kaluza, Sitzungsber Preuss Akad Wiss. Berlin. (Math. Phys.), 996, (1961), O. Klein, Z. Phys. 37, 895, (1926).
- [2] J. Ponce de Leon and P. S. Wesson. Journal of Math. Phys. 34.9, 4080-4092, (1993).

- [3] O. Klein, Zeits. Phys. 37, 895, (1926).
- [4] J. M. Overduin, and Paul S. Wesson. Physics Reports 283.5, 303, (1997).
- [5] O. Heckmann, P. Jordan and W. Fricke, Z. Astrophys. 28, 113, (1951).
- [6] P. S. Wesson, Phys. Lett, 276B, 299, (1992).
- [7] D. J. Gross and M. J. Perry, Nucl. Phys. B, 226, 29, (1983).
- [8] A. Davidson and D. Owen, Phys. Lett. 155B, 247, (1985).
- [9] T. Ortin, Gravity and String, Cambridge University Press, (2004).
- [10] P. S. Wesson and J. Ponce de Leon. Astronomy and Astrophysics 294, 1, (1995).
- [11] P. S. Wesson and J. Ponce de Leon. Journal of Math. Phys. 33.11, 3883, (1992).
- [12] G. Lessner, Phys. Rev. D25, 3202, (1982).
- [13] Y. Thiry, Comptes Rendus de la Academie des Sciences (Paris) 226, 216, (1948).
- [14] D. Baleanu, S. Codoban. General Relativity and Gravitation, 31(4), 497, (1999).
- [15] D. A. Konkowski, T. M. Helliwell, L. C. Shepley, Physical Review D, 31(6), 1178, (1985).
- [16] J. B. Griffiths and J. Podolsky, Exact Space-Times in Einstein's General Relativity, Cambridge University Press, (2009).
- [17] R. M. Wald, General Relativity, Chicago: University of Chicago Pres, (1984).
- [18] M. Alfredo, and T. Matos. Classical and Quantum Gravity 13, no. 3, 345, (1996).
- [19] J. W. Van Holten. Physics Letters B 342.1 47-52, (1995).
- [20] R. D. Sorkin. Physical Review Letters 51.2, 87, (1983). R. D. Sorkin, Phys. Rev. Lett, 51, 87, (1983).
- [21] Paul S. Wesson and J. Ponce de Leon. Classical and Quantum Gravity 11.5, 1341, (1994).
- [22] J. L. Jaramillo and E.ourgoulhon. Mass and angular momentum in general relativity. In L. Blanchet, A. Spallicci, and B. Whiting, editors, Mass and Motion in General Relativity, volume 162 of Fundamental Theories of Physics, pages 87-124. Springer Netherlands, (2011).
- [23] G.W. Gibbons and N.S. Manton, Nucl. Phys. B274, 183, (1986).